A Natural Entanglement Test

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We derive a simple lower bound on the geometric measure of entanglement for mixed quantum states in the case of a general multipartite system. Obtained bound leads to *natural entanglement criteria* with a straightforward interpretation. The main ingredient of the presented derivation is the triangle inequality applied to the root infidelity distance in the space of density matrices. Proposed criteria lead in the two–qubit case to an experimentally accessible, powerful entanglement test, faithful also for a family of full rank mixed states.

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Quantum entanglement characterizes non–classical correlations in a quantum system consisting of several subsystems. In the case of a pure quantum state, any correlations between subsystems, that can be detected in coincidence experiments, confirm entanglement. However, in any realistic experiment one has to cope with mixed quantum states, for which the problem becomes more involved, as quantum and classical correlations may exist. To detect reliably quantum entanglement for a mixed quantum state one needs to rule out the more common case of classical correlations.

While efficient detection of quantum entanglement is a not easy task in quantum information theory, it is more difficult to characterize this phenomenon quantitatively basing on the results of partial measurements that do not allow for full state reconstruction. Known schemes of such experimental procedure require interactions between many copies of the state investigated [1]. Basing on interaction between two copies of the state one can estimate a lower bound on an entanglement measure [2] in terms of a two–copy entanglement witness that reproduces the difference between global and local entropy [3]. However this measurable lower bound does not provide a faithful entanglement test (sharp if and only if type criteria) in the case of mixed states of a full rank.

In this work we build on a pragmatic approach advocated in [4, 5], in which one attempts to construct entanglement measures accessible in an experiment. We derive a lower bound for the geometric measure of entanglement [6, 7] valid for an arbitrary mixed quantum state and propose a concrete experimental setup capable to measure this quantity in the simplest case of a two–qubit system. The *natural entanglement test* proposed in this paper occurs to be faithful for families of mixed states constructed experimentally in laboratories.

Consider an arbitrary multipartite quantum system described in the Hilbert space $\mathcal{H}=\bigotimes_I \mathcal{H}^I$ with no assumption about the dimensionality of the particular subspace $\mathcal{H}^I$ representing the Ith subsystem. We shall denote by $\mathcal{S}$ the set of mixed, separable states $\sigma_{\rm sep}$, while by $\mathcal{S}_1$ the set of separable states of rank one — pure separable (product) states $|\phi\rangle_{\rm sep}=\otimes_I|\phi_I\rangle$. In principle, we will discuss the case of complete separability (as shown by the example state $|\phi\rangle_{\rm sep}$), however our approach extends in a natural way to capture other types of entanglement, like genuine entanglement (being non bi–separable), appearing in systems involving more than two parties. Since the dimensions of all subsystems can be arbitrarily chosen it is sufficient to invoke another (relevant for the specific type of entanglement) partition of $\mathcal{H}$.

In our considerations we shall use the root infidelity distance between two mixed states ρ_1 and ρ_2 [8]:

$$C_F(\rho_1,\rho_2)=\sqrt{1-F(\rho_1,\rho_2)},\tag{1}$$

defined with the help of the fidelity $F(\rho_1,\rho_2)$. To derive our result we will however only use the fidelity involving at least one pure state, thus we need only the restricted, simpler formula for the fidelity

$$F(\rho,|\Psi\rangle\langle\Psi|)=\langle\Psi|\rho|\Psi\rangle.\tag{2}$$

From now on we shall simplify the notation and always write the argument $|\Psi\rangle\langle\Psi|$ as $|\Psi\rangle$, for example $F(\rho,|\Psi\rangle\langle\Psi|)\equiv F(\rho,|\Psi\rangle)$.

The last quantity we need to introduce is the geometric measure of entanglement [6, 7], which in the case of pure states is defined as:

$$\begin{aligned}E(|\Psi\rangle) &= 1-\max_{|\phi\rangle\in\mathcal{S}_1}|\langle\phi|\Psi\rangle|^2 \\ &\equiv \min_{|\phi\rangle\in\mathcal{S}_1}C_F^2(|\phi\rangle,|\Psi\rangle).\end{aligned}\tag{3}$$

The second, equivalent definition follows directly from Eqs. (1, 2). The operational interpretation of the measure $E(|\Psi\rangle)$ is straightforward. If the state $|\Psi\rangle$ is separable it belongs to the set $\mathcal{S}_1$, thus the minimal infidelity

distance is 0 since one can always chose $|\phi\rangle \in \mathcal{S}_1$ to be equal $|\Psi\rangle$. Oppositely, if $|\Psi\rangle$ is entangled, then its distance from the set $\mathcal{S}_1$ is always nonzero.

The geometric measure of entanglement for mixed states is defined [7] with the help of the convex roof construction:

$$E(\rho) = \min_{\mathcal{E}} \sum_i p_i \min_{|\phi\rangle \in \mathcal{S}_1} C_F^2(|\phi\rangle, |\Psi_i\rangle), \tag{4}$$

where the ensemble $\mathcal{E} = \{p_i, |\Psi_i\rangle\}$ represents the mixed state ρ, i.e. $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. Surprisingly, it was shown [9] that $E(\rho)$ is simultaneously a distance measure $E(\rho) = \min_{\sigma\in\mathcal{S}} C_F^2(\sigma, \rho)$.

Natural entanglement criteria.— Any density matrix representing a bi–partite system can be characterized by its *product numerical radius* $L(\rho)$, often used in the theory of quantum information [10]. This quantity can be defined as the maximal expectation value of ρ among normalized pure product states,

$$L(\rho) = \max_{|\phi\rangle \in \mathcal{S}_1} \langle\phi|\rho|\phi\rangle. \tag{5}$$

Note that $E(|\Psi\rangle) \equiv 1 - L(|\Psi\rangle)$. In the case of pure states the definition (3) can be rewritten to the form

$$(L(|\Psi\rangle) < 1) \Leftrightarrow (|\Psi\rangle \text{ - entangled}), \tag{6}$$

which in fact establishes the property of being entangled (it gives a necessary and sufficient condition). The above logical sentence cannot be generalized to the case of mixed states because the Cauchy–Schwarz inequality for Hermitian matrices tells us that $L(\rho) \le \sqrt{\mathrm{Tr}\rho^2}$. The upper bound $\sqrt{\mathrm{Tr}\rho^2}$ contains no information about entanglement of ρ, however as it quantifies the mixedness of ρ it can be substantially less than 1. Moreover, it can be saturated only for pure separable states, so in general the condition $L(\rho) < \sqrt{\mathrm{Tr}\rho^2}$ does not capture the entanglement of ρ but rather its mixedness. According to the above discussion we find it not obvious that the following entanglement criteria hold:

$$\left(L(\rho) < \mathrm{Tr}\rho^2\right) \Rightarrow (\rho \text{ - entangled}). \tag{7}$$

Due to the strong similarity of (7) with the statement (6) we shall call the criteria (7) as the *natural entanglement criteria*, since (6) is nothing else than the natural description of pure–states entanglement. It is important to point out that the symbol $\Leftrightarrow$ from (6) is in (7) replaced by $\Rightarrow$, so that (7) cannot be regarded as an equivalent definition of non–separability (it gives only a sufficient condition).

The natural entanglement criteria have been recently recognized in [11, 12]. In [11] they appear in the form of a nonlinear entanglement witness $L(\rho)\mathbb{1} - \rho$, while in [12] a more general criteria (see Eq. (12) from [12]) that involve (7) as a special case, have been introduced. In our considerations the criteria (7) come as an immediate consequence of the following lower bound for the square root of the geometric measure of entanglement

$$\sqrt{E(\rho)} \ge R(\rho) = \sqrt{1 - L(\rho)} - \sqrt{1 - \mathrm{Tr}\rho^2}, \tag{8}$$

which will be derived in the next paragraph. This situation is similar to the case of *purity/entropy* entanglement criteria [13] given in terms of the Rényi entropy H_α, which in the special case $\alpha = 2$ was shown [2] to establish the lower bound for the concurrence [14]. From (8) one can obviously find the lower bound for $E(\rho)$, which reads $(\max[R(\rho); 0])^2$. It is important to take the maximum first, in order to avoid cases when negative values of $R(\rho)$ can give a positive, unphysical contribution $R^2(\rho)$ to the lower bound of $E(\rho)$.

We start the derivation of (8) with an arbitrary expansion $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$ of the mixed state ρ. For some fixed index i we chose a pure state $|\Psi_i\rangle$, and another pure state $|\phi\rangle$ to be specified. Since the root infidelity (1) is a legitimate metric we can write down the triangle inequality for $C_F(|\phi\rangle, \rho)$ with $|\Psi_i\rangle\langle\Psi_i|$ as a third state:

$$C_F(|\phi\rangle, \rho) \le C_F(|\phi\rangle, |\Psi_i\rangle) + C_F(|\Psi_i\rangle, \rho). \tag{9}$$

If we next take the minimum with respect to $|\phi\rangle \in \mathcal{S}_1$ and use the definitions (1, 3, 5) we obtain

$$\sqrt{1 - L(\rho)} \le \sqrt{E(|\Psi_i\rangle)} + C_F(|\Psi_i\rangle, \rho). \tag{10}$$

In the next step we shall multiply the resulting inequality by p_i and sum over i. The term $\sqrt{1 - L(\rho)}$ is independent of i, while for the two terms on the right hand side we shall apply the following estimates originating from the concavity of the $\sqrt{\cdot}$ function:

$$\sum_i p_i \sqrt{E(|\Psi_i\rangle)} \le \sqrt{\sum_i p_i E(|\Psi_i\rangle)}, \tag{11}$$

$$\sum_i p_i C_F(|\Psi_i\rangle, \rho) \le \sqrt{1 - \sum_i p_i \langle\Psi_i|\rho|\Psi_i\rangle}. \tag{12}$$

In the final step we shall recognize that the sum over i on the right hand side of (12) is equal to $\mathrm{Tr}\rho^2$, so that is independent of the given ensemble $\mathcal{E} = \{p_i, |\Psi_i\rangle\}$. This implies that we can immediately minimize with respect to $\mathcal{E} = \{p_i, |\Psi_i\rangle\}$ producing on the right hand side of (11) the quantity $\sqrt{E(\rho)}$. When we apply the above estimates to Eq. (10) we obtain after a one–step rearrangement, the desired lower bound (8).

Natural entanglement test.— In the previous paragraph we derived the lower bound (8) for the geometric measure of entanglement in a general case of a mixed state ρ acting on the multipartite Hilbert space $\mathcal{H}$. Our aim is now to allow the bound $R(\rho)$ to be experimentally accessible, at least in the case of two qubits. To

this end let us represent the Hilbert space $\mathcal{H}$ as a tensor product $\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\bar{\mathcal{A}}}$, where the subsystem „$\mathcal{A}$" shall be considered as a one particle system while „$\bar{\mathcal{A}}$" might be treated as a remaining part. We represent $|\phi\rangle \in \mathcal{S}_1$ as a bi–partite tensor product $|\phi\rangle = |\phi_{\mathcal{A}}\rangle \otimes |\phi_{\bar{\mathcal{A}}}\rangle$, where $|\phi_{\bar{\mathcal{A}}}\rangle \in \mathcal{H}_{\bar{\mathcal{A}}}$ shall be, as a potentially multipartite state, also separable.

We can rewrite the definition (5) to the form:

$$L(\rho) = \max_{|\phi_{\bar{\mathcal{A}}}\rangle} \max_{|\phi_{\mathcal{A}}\rangle} \langle \phi_{\mathcal{A}} | \Omega | \phi_{\mathcal{A}} \rangle, \tag{13}$$

where the positive semi–definite operator Ω acting on $\mathcal{H}_{\mathcal{A}}$ is given by the formula

$$\Omega = \mathrm{Tr}_{\bar{\mathcal{A}}} \left(\rho \mathbb{1}_{\mathcal{A}} \otimes |\phi_{\bar{\mathcal{A}}}\rangle \langle \phi_{\bar{\mathcal{A}}}| \right). \tag{14}$$

Using the above representation of $L(\rho)$ we are able to perform the maximization with respect to $|\phi_{\mathcal{A}}\rangle$, since $\max_{|\phi_{\mathcal{A}}\rangle} \langle \phi_{\mathcal{A}} | \Omega | \phi_{\mathcal{A}} \rangle = \omega_0$, where $\omega_0 \equiv \omega_0(\rho, |\phi_{\bar{\mathcal{A}}}\rangle)$ denotes the largest eigenvalue of Ω.

In an equivalent form we can write $L(\rho) = \max_{|\phi_{\bar{\mathcal{A}}}\rangle} \|\Omega\|_\infty$ where the q–norm ($q > 0$) of an operator is defined as $\|\Omega\|_q = (\mathrm{Tr}\Omega^q)^{1/q}$. An important property of q–norms is that $\|\Omega\|_q \geq \|\Omega\|_\infty$, thus we obtain a family of lower bounds $R_q(\rho)$ given by the right hand side of (8) with $L(\rho)$ substituted by $L_q(\rho) = \max_{|\phi_{\bar{\mathcal{A}}}\rangle} \|\Omega\|_q$. For $p \leq q$ we have $R_p(\rho) \leq R_q(\rho)$, so that the original bound $R(\rho) \equiv R_\infty(\rho)$ is the sharpest, however bounds with different values of q, for example $q = 2$ might for a particular state ρ become easier to compute and measure.

We shall now restrict our considerations to the case of a two–qubit case , i.e. $\mathcal{A}$ and $\bar{\mathcal{A}}$ both represent single qubit spaces. In that particular case Ω is a 2×2 matrix, so that ω_0 is determined by $\mathrm{Tr}\Omega$ and $\mathrm{Tr}\Omega^2$:

$$\omega_0\left(\mathrm{Tr}\Omega, \mathrm{Tr}\Omega^2\right) = \frac{\mathrm{Tr}\Omega + \sqrt{2\mathrm{Tr}\Omega^2 - (\mathrm{Tr}\Omega)^2}}{2}. \tag{15}$$

For a given one–qubit pure state $|\phi_{\bar{\mathcal{A}}}\rangle$ both parameters $\mathrm{Tr}\Omega$ and $\mathrm{Tr}\Omega^2$ can be measured in a single experiment [15] based on the Hong–Ou–Mandel (H–O–M) interferometry [16]. The relevant experimental setup can be combined with the measurement of purity $\mathrm{Tr}\rho^2$ performed [17] with the help of two H–O–M interferences. Thus, we obtain experimentally accessible entanglement test relevant for an arbitrary two–qubit state ρ,

$$\left(\max_{|\phi_{\bar{\mathcal{A}}}\rangle} \omega_0\left(\mathrm{Tr}\Omega, \mathrm{Tr}\Omega^2\right) < \mathrm{Tr}\rho^2 \right) \Rightarrow (\rho \text{ - entangled}), \tag{16}$$

which, as a particular case of (7), we shall call the *natural entanglement test*. In fact, the above result leads in the case of two qubits to the experimentally accessible lower bound for the geometric measure of entanglement.

The Werner state.— Let us investigate an example of the generalized Werner state ($0 \leq p \leq 1$):

$$\rho_w(p, \lambda) = (1-p)\, |\Psi_\lambda\rangle \langle \Psi_\lambda| + p \frac{I \otimes I}{4}, \tag{17}$$

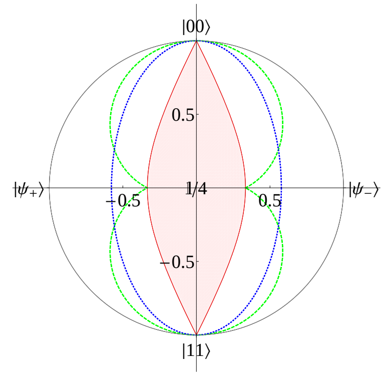

Figure 1: (color online). The plane of ρ_w (points inside gray circle) in the polar coordinate $r = 1 - p$, $\theta = 2 \arccos \sqrt{\lambda}$. States in the shaded region bounded by solid red curve are separable (PPT). Entangled states outside dashed green curve and outside dotted blue curve are detected by the natural entanglement test and the purity test respectively.

where $|\Psi_\lambda\rangle = \sqrt{\lambda}\, |00\rangle + \sqrt{1-\lambda}\, |11\rangle$. One can straightforwardly calculate

$$\mathrm{Tr}\rho_w^2 = 1 - \frac{3p}{2} + \frac{3p^2}{4}, \quad L(\rho_w) = \frac{p}{4} + (1-p) \max[\lambda; 1-\lambda]. \tag{18}$$

From the PPT criteria we know that ρ_w is separable when $p \geq 1 - (1 + 4\eta)^{-1}$ for $\eta = \sqrt{\lambda(1-\lambda)}$. In a particular case when $|\Psi_\lambda\rangle$ is maximally entangled, i.e. $\lambda = 1/2$ the threshold for separability is thus $p_{\mathrm{sep}} = 2/3$. According to the natural entanglement test the state ρ_w is entangled for $p \leq 4(1 - \max[\lambda; 1-\lambda])/3$. Note that for $\lambda = 1/2$ we obtain the separability threshold $p_{\mathrm{nat}} = 2/3$, so that a full range of entangled states is detected. The power of the test is thus most remarkable for the original Werner state. Its rotations for arbitrary local unitaries U_1, U_2

$$\rho_w' = U_1 \otimes U_2 \rho_w \left(p, \frac{1}{2}\right) U_1^\dagger \otimes U_2^\dagger, \tag{19}$$

becomes the most general mixture of maximally entangled two–qubit state and the uniform noise. The natural entanglement test provides a faithful lower bound of entanglement measure for *all* members of the family (19).

In Fig. 1 we compare the natural entanglement test (dashed green curve) and the purity/entropy test [13] (dotted blue curve) given by the condition $\mathrm{Tr}\rho_{A/B}^2 \geq \mathrm{Tr}\rho^2$ satisfied when ρ is separable. Here $\rho_{A/B}$ denote density operators of single subsystem A and B respectively. For $|\Psi_\lambda\rangle$ in the neighborhood of the maximally entangled state ($\lambda = 1/2$) the natural entanglement test (16) outperforms the purity test.

Experimental implementation.— In fact, there shall be no competition between the purity/entropy test and the natural entanglement test since both tests can be simultaneously performed. The experimental setup suitable for this task (see Fig. 2) is based on the scheme

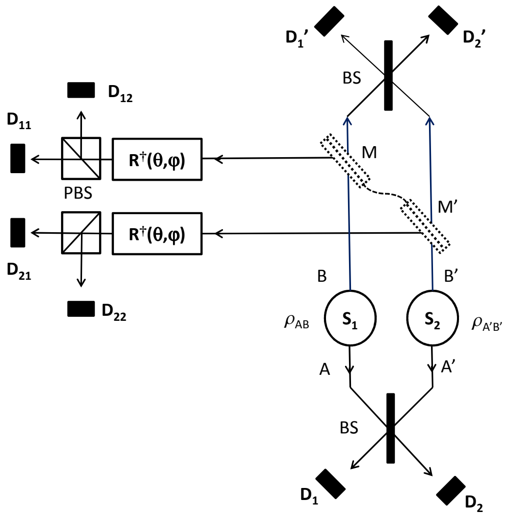

Figure 2: Entanglement test via locally enhanced double purity checker. Experimental setup designed for simultaneous measurement of the purity/entropy test [13, 17] and the natural entanglement test (16).

with two H–O–M interferometers [17] used for the entanglement test via purity analysis. Two perfectly reflecting mirrors M and M′ are randomly put in and out in a correlated way to switch into the setup that provides the estimation of the collectibility [18]. The combination of statistics from the two settings gives the value of the powerful nonlinear entanglement witness that surprisingly detects entanglement of all two–qubit mixtures of maximally entangled state with a uniform noise (19). Note that the latter class is a very popular description of any source of entanglement in the case when only visibility is known.

Since the part of the setup Fig. 2 responsible for purity measurements has been described in [17], we shall briefly answer the question how to infer the values of parameters $\mathrm{Tr}\Omega$ and $\mathrm{Tr}\Omega^2$ with the help of the „collectibility part" [15] of the setup. Two polarization rotators $R^{\dagger}\left(\theta,\varphi\right)$ in the same setting represent the state $|\phi_{\bar{A}}\rangle$, thus the optimization with respect to both angles is necessary. After $R^{\dagger}\left(\theta,\varphi\right)$ the photon B may be captured by the detector D_{11} (with probability of click p_+), or by D_{12} (with probability p_-), leaving the photon A in the state σ_+ or σ_- respectively. Similarly the photon B′ may be captured by D_{21} (with the same probability p_+), or by D_{22} (probability p_-), leaving the photon A′ from the second copy in the state σ_+ or σ_-. The bottom H–O–M interferometer takes the input states of the photons A and A′ reproducing the values $\mathrm{Tr}\sigma_+^2$, $\mathrm{Tr}\sigma_-^2$ or $\mathrm{Tr}\left(\sigma_+\sigma_-\right)$, depending on the case. All possibilities are summarized in Table I.

For a given state $|\phi_{\bar{A}}\rangle$ — a fixed choice of $R^{\dagger}\left(\theta,\varphi\right)$ — we can represent the parameters $\mathrm{Tr}\Omega$ and $\mathrm{Tr}\Omega^2$ in two different ways (labeled by "±") as:

$$\mathrm{Tr}\Omega = p_{\pm}, \qquad \mathrm{Tr}\Omega^2 = p_{\pm}^2\mathrm{Tr}\sigma_{\pm}^2. \tag{20}$$

click B	click B′	state of A	state of A′	the bottom H–O–M
D_{11}; p_+	D_{21}; p_+	σ_+	σ_+	$\mathrm{Tr}\sigma_+^2$
D_{12}; p_-	D_{22}; p_-	σ_-	σ_-	$\mathrm{Tr}\sigma_-^2$
D_{11}; p_+	D_{22}; p_-	σ_+	σ_-	$\mathrm{Tr}\left(\sigma_+\sigma_-\right)$
D_{12}; p_-	D_{21}; p_+	σ_-	σ_+	$\mathrm{Tr}\left(\sigma_+\sigma_-\right)$

Table I: Summary of the „collectibility" part of the setup.

Using this result we can build the eigenvalue ω_0 (Eq. 15). After optimization with respect to (θ,φ) the two sign cases from Eq. (20) shall give the same value.

Let us emphasize in passing that the criteria (7) are strong enough to detect bound entanglement. A concrete family of PPT entangled states of a 3×3 system will be discussed elsewhere.

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